

A Note on Finding Dual Feedback Vertex Set

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Abstract. For an edge-bicolored graph G where each edge is colored either red or blue, a vertex set S is a dual feedback vertex set if S hits all blue cycles and red cycles of G . In this paper, we show that a dual feedback vertex set of size at most k can be found in time $O^*(c_1^k)$ and all minimal dual feedback vertex set of size at most k can be enumerated in time $O^*(c_2^{k^2+k})$ by compact representations for constants c_1 and c_2 .

Keywords: edge-bicolored graph, dual feedback vertex set, parameterized complexity.

1 Introduction

Given an undirected graph G , the FEEDBACK VERTEX SET problem (FVS) asks if there is a vertex set of size at most k that hits all cycles of G . FVS is one of the most fundamental graph problems whose NP-completeness is proved by Karp [15]. FVS has been extensively studied under the framework of parameterized complexity [12]. Since Downey and Fellows [11], and Bodlaender [3] gave the first FPT algorithms of FVS, there is a long race for the fastest FPT algorithm [12,18,14,19,13,10,7,6,16]. Besides the classical feedback vertex set problem, many interesting generalizations of FVS have been studied. For example, SUBSET FEEDBACK VERTEX SET, which requires to hit all cycles going through some specified vertices, is shown to be FPT [21,17]. GROUP FEEDBACK VERTEX SET, a more generalized problem, is studied and proved to be FPT by Cygan et al. [9], where each edge is labeled with some group element, and it requires to hit all cycles that evaluate to a non-null element.

In this paper, we focus on edge-bicolored graphs that are simple undirected graphs with each edge colored either blue or red, and consider the parameterized complexity of the following generalization of FVS:

DUAL FEEDBACK VERTEX SET (DFVS):

INPUT: Edge-bicolored graph G , parameter k .

QUESTION: Find a vertex set S of size at most k that hits all blue and red cycles.

Recently, Cai and Ye [5] have systematically considered such kind of vertex deletion problems on edge-bicolored graphs, i.e., problems of deleting vertices on an edge-bicolored graph to make the blue and red graphs satisfy some

properties respectively. They give a complete characterization of the INDUCED (Π_b, Π_r) -SUBGRAPHS problems for hereditary properties Π_b and Π_r , which finds, on an edge-bicolored graph, an induced subgraph of k vertices whose blue and red graphs satisfy the properties Π_b and Π_r respectively. Their results include the W[1]-hardness of the parametric dual of DFVS, i.e., INDUCED (Π_b, Π_r) -SUBGRAPHS for $\Pi_b = \Pi_r$ being acyclic.

In this paper, we give an FPT algorithm for DFVS, and show that all minimal solutions of size at most k can be enumerated in FPT time by compact representations. In connection with these results, we generalize the FPT algorithm of DFVS to show that, given an edge-colored graph G with h colors, it takes time $O^*(c^{hk \log k})$ ¹ to find a vertex set of size at most k that hits all monochromatic cycles of G .

For a graph G , $V(G)$ and $E(G)$ denote its vertex set and edge set respectively, and n and m , respectively, are numbers of vertices and edges of G . For a subset $V' \subseteq V(G)$, $N_G(V')$ denotes the neighbors of V' in $V(G) - V'$ and $G[V']$ the subgraph of G induced by V' .

For an edge-bicolored graph $G = (V, E_b \cup E_r)$, $G_b = (V, E_b)$ and $G_r = (V, E_r)$, respectively, denote the blue graph and red graph of G . For a vertex $v \in V$, $d_b(v)$ and $d_r(v)$ denote the degree of v in G_b and G_r respectively. We use $d(v) = (d_1, d_2)$ to denote the degree of v in G , where $d_1 = d_b(v)$ and $d_2 = d_r(v)$. A subset $V' \subseteq V(G)$ is a *dual feedback vertex set* of G if both $G_b - V'$ and $G_r - V'$ are acyclic.

An instance (I, k) of a parameterized problem Π consists of two parts: an input I and a parameter k . We say that a parameterized problem Π is fixed parameter tractable (FPT) if there is an algorithm solving every instance (I, k) in time $f(k)|I|^{O(1)}$ for some computable function f .

In the rest of the paper, we present FPT algorithms for DFVS in Section 2, and show how to enumerate minimal solutions of DFVS in Section 3. We conclude with discussions in Section 4.

2 Dual Feedback Vertex Set

In this section, we show that DFVS can be solved in time $O^*(c^k)$ following the result of Guo et al.'s [13] which enumerates all minimal feedback vertex sets in FPT time by compact representations. A compact representation C is a collection of pairwise disjoint vertex sets. A vertex set S that contains exactly one element of every set of a compact representation C is a minimal dual feedback set of size at most k , and we say that S is represented by C .

Theorem 1. *DFVS can be solved in time $O^*(c^k)$ for a constant c .*

Proof. We first enumerate all compact representations of minimal feedback vertex set of size at most k for both G_b and G_r in time $O^*(c_1^k)$ for a constant c_1 by Guo et al.'s algorithm [13]. Since a minimal dual feedback vertex set is always

¹ The O^* -notation suppresses factors polynomial in the input size.

a union of a minimal feedback vertex set of G_b and a minimal feedback vertex set of G_r , a minimal solution of DFVS is a minimal vertex set that covers every vertex set in $C_b \cup C_r$ for two compact representations C_b of G_b and C_r of G_r . Given two compact representations C_b and C_r of G_b and G_r respectively, we show that a minimum vertex set S that intersects every vertex set in $C_b \cup C_r$ can be found in polynomial time.

To this end, we build an auxiliary graph H as following:

- For a vertex set $S_i \in C_b \cup C_r$, create a vertex s_i .
- If a vertex v is in both $S_i \in C_b$ and $S_j \in C_r$, add an edge $e_v = s_i s_j$.
- If there are t parallel edges between a pair of vertices, select one edge to represent the t edges and delete other $t - 1$ edges.

Note that each edge represents the intersection of two sets, thus if there are t parallel edges we can arbitrary delete $t - 1$ edges. Since C_b , also C_r , contains at most k vertex disjoint sets, H has at most $2k$ vertices and k^2 edges. For each isolated vertex s_i of H , we arbitrary put a vertex $v \in S_i$ into S . Then we find a minimum edge set F that covers all non-isolated vertices, which can be done in time $O(k^3)$ following the Theorem 3.7 in Cai's paper [4]. For an edge $e_v \in F$, put the vertex v into S . Then S is a minimum vertex set that covers all sets of $C_b \cup C_r$.

As there are c_1^k ways to select a compact representation C_b , also C_r , the running time for this algorithm is $O^*(c_1^{2k}) = O^*(c^k)$ for a constant c . ■

One crucial step of the above algorithm is to find a minimal set that covers two compact representations. We observe that the analogous step can be done in FPT time for MULTI-FEEDBACK VERTEX SET (MFVS) which is the problem of finding a vertex set of size at most k that hits all monochromatic cycles on an edge-colored graph. Based on the observation, we can establish the fixed parameter tractability of MFVS.

Theorem 2. *MFVS can be solved in time $O^*(c^{hk \log k})$ where h is the number of colors of the input edge-colored graph.*

Proof. Let G be an edge-colored graph with h colors, and G_1, \dots, G_h be the monochromatic subgraphs of G . We first obtain compact representations of minimal feedback vertex set of size at most k for G_i by the algorithm of Guo et al.'s [13], which takes time $O^*(hc_1^k)$ for a constant c_1 .

Let C_i be a compact representation of G_i . Since a minimal solution of MULTI-FEEDBACK VERTEX SET is always a union of minimal feedback sets of G_i , a vertex set of size at most k that intersects all sets of C_i for $1 \leq i \leq h$ will be a solution. To find such a vertex set, we build an auxiliary graph H as following:

- For each vertex set $S_i \in C_1 \cup \dots \cup C_h$, create a *set-vertex* s_i .
- For each vertex v of G , create a *copy-vertex* v' and add edge $v' s_i$ if $v \in S_i$.
- Create an edge $v^* u^*$, and add all edges between v^* and copy-vertices.

It is easy to check that H has a dominating set of size at most $k+1$ iff there is a set of at most k copy-vertices that dominates all set-vertices, and thus corresponds to a solution of MFVS. Furthermore, given a dominating set S of size at most $k+1$ of H , we can obtain a set of copy-vertices which corresponds to a solution of MFVS by replacing each set-vertex $s_i \in S - \{v^*, u^*\}$ by a copy-vertex in $N(s_i)$.

Since the vertex sets are pairwise disjoint in a compact representation, each copy-vertex is adjacent to at most one set-vertex of a compact representation. Each copy-vertex has degree at most $h+1$, and thus H is a $(h+1)$ -degenerated bipartite graph. Then we can apply the algorithm of Alon and Gutner [1], which finds a dominating set of size at most l on a d -degenerated graph in time $l^{O(dl)}n$, to obtain a dominating set of size at most $k+1$ in time $(k+1)^{O((h+1)(k+1))}n = O(c_2^{hk \log k} n)$ for a constant c_2 . Therefore, given one compact representation of each G_i , we can find a solution among these compact representations in time $O(c_2^{hk \log k} n)$.

Since there are c_1^k ways to choose a compact representation C_i of G_i , the total running time of the algorithm is $O^*(c_1^k c_2^{hk \log k}) = O^*(c^{hk \log k})$ for a constant c . ■

3 Dual Feedback Vertex Set Enumeration

In this section we give two FPT algorithms to enumerate all minimal dual feedback vertex set of size at most k by compact representations.

We first introduce the algorithm based on the proof of Theorem 1. In the auxiliary graph H of Theorem 1, a vertex s_i represents a set S_i and an edge $s_i s_j$ represents the vertex set $S_i \cap S_j$. We need to cover all vertices of H by selecting at most k edges or vertices. Since H contains at most $2k$ vertices and k^2 edges, we can enumerate all minimal sets of at most k elements that covers all vertices of H in time $O^*(2^{k^2+2k})$ by exhausted search. Each minimal set is a compact representation of minimal dual feedback vertex sets of the original edge-bicolored graph G . And the running time is $O^*(2^{k^2+2k} c_1^{2k}) = O^*(c^{k^2+k})$ for a constant c .

Next we present another FPT algorithm which follows the idea of iterative compression introduced by Reed, Smith and Vetta [20]. The main portion is to solve the following problem:

Problem 1. Given an edge-bicolored graph $G = (V, E_b \cup E_r)$ and a dual feedback vertex set S , find a dual feedback vertex set S' such that $S' \cap S = \emptyset$ and $|S'| < |S|$.

We show that all minimal solutions of Problem 1 can be enumerated by compact representations in FPT time. To this end, we introduce some simple reduction rules.

Rule 1. If there is a vertex v with $d(v) = (0, 0)$, reduce to $G = G - v$.

Rule 2. If there is a vertex v with $d_b(v) = 1$ (resp., $d_r(v) = 1$), remove the blue (resp., red) edge incident to v .

Rule 3. For vertices $v \notin S$ and $x, y \in S$ such that x and y belong to the same blue (resp., red) component of $G[S]$, and there are two blue (resp., red) edges vx and vy , reduce to $G = G - v$ and add v to S' .

Above three rules are obviously safe, and can be executed in polynomial time. Call an edge-bicolored graph *reduced graph* if none of above rules is applicable. Then in a reduced graph, there are no vertex $v \notin S$ such that $d(v) = (0, 0)$, $(1, *)$ or $(*, 1)$. In a reduced graph G , let $V_{>2}^b$ (resp., $V_{>2}^r$) be the set of vertices in $V(G) - S$ whose blue (resp., red) degree is larger than 2, $V_{\leq 2}$ be the set of vertices whose both blue degree and red degree are at most 2, i.e., $V_{\leq 2} = V(G) - S - V_{>2}^b - V_{>2}^r$. A blue path $P = \{v_1, v_2, \dots, v_p\}$ is a *maximal blue path* if $v_i \notin S$, $d_b(v_i) = 2$ for $1 \leq i \leq p$ and both v_1 and v_p are adjacent to vertices in $S \cup V_{>2}^b$. A *maximal red path* is defined similarly.

Now we introduce two rules that reduce the size of a maximal blue (red) path in a reduced graph.

Rule 4. Let P_b (resp., P_r) be a maximal blue (resp., red) path, and V^* be the vertex set containing all vertices of P_b (resp., P_r) with degree $(2, 0)$ (resp., $(0, 2)$). If $|V^*| \geq 2$, select one vertex $v \in V^*$ to represent V^* . For other vertices $u \in V^*$, remove u and add a blue (resp., red) edge between u 's two blue (resp., red) neighbors.

Rule 5. Let P_b and P_r be a maximal blue path and red path respectively. If P_b and P_r share common vertices, select one common vertex v to represent all common vertices, i.e., $V(P_b) \cap V(P_r)$. For other vertices $u \in V(P_b) \cap V(P_r)$, remove u and add a blue (resp., red) edge between u 's two blue (resp., red) neighbors.

Lemma 1. Rule 4 and Rule 5 are safe. Let m_b and m_r , respectively, be the numbers of maximal blue paths and maximal red paths. After applying Rule 4 and Rule 5, $|V_{\leq 2}| \leq (m_b + 1) * (m_r + 1)$.

Proof. Suppose that v and u are both on a maximal blue path P_b and a maximal red path P_r . Since all monochromatic cycles going through vertex u also go through v , if a solution contains u , we can replace u with v . A minimal solution may contain at most one vertex from $V(P_b) \cap V(P_r)$, and if it does contain one such vertex, it doesn't matter which vertex it contains. Thus we can use one arbitrary vertex to represent $V(P_b) \cap V(P_r)$ and dissolve other vertices. Therefore, Rule 5 is safe. Similarly, Rule 4 is safe. Note that Rule 4 and Rule 5 can be executed in polynomial time.

Now consider a maximal blue path P_b . We bound the number of vertices in $V(P_b) \cap V_{\leq 2}$. By Rule 4, P_b contains at most one vertex with degree $(2, 0)$. By Rule 2, P_b contains no vertices with degree $(2, 1)$. Then we consider vertices of P_b with degree $(2, 2)$. Note that each such vertex is an intersection of P_b and a maximal red path. By Rule 5, the number of such vertices is at most m_r . Therefore $|V(P_b) \cap V_{\leq 2}| \leq 1 + m_r$. Similarly, for a maximal red path P_r , $|V(P_r) \cap V_{\leq 2}| \leq 1 + m_b$. Since each vertex in $V_{\leq 2}$ is on a maximal blue (red)

path, we have $|V_{\leq 2}| \leq (m_b + 1) * (m_r + 1)$. ■

Let us focus on the blue (red) subgraph of an reduced graph. Guo et al. show that there are at most $14|S|$ vertices with degree larger than 2 and at most $16|S|$ maximal blue (red) paths in the proof of Lemma 2 and Lemma 6 in [13]. Relying on these results, we enumerate all minimal solutions of Problem 1 in FPT time.

Lemma 2. *All minimal solutions of Problem 1 can be enumerated in $O^*(c^{|S|^2+|S|})$ time for a constant c by compact representations.*

Proof. Reduce the input edge-bicolored graph G by Rule 1 to Rule 5.

We first determine which vertices of $V_{>2} = V_{>2}^b \cup V_{>2}^r$ of G are in a minimal solution. By the proof of Lemma 2 in [13], we have $|V_{>2}| \leq 28|S|$. We guess which vertices of $V_{>2}$ are in a solution S' and branch into $\sum_{t=0}^{|S|-1} \binom{28|S|}{t}$ cases. In each case, we put t one-element sets into C , delete these vertices in G and apply Rule 1 to Rule 5 to further reduce the graph. Denote by G' the resulting new edge-bicolored graph.

Then we determine, for each case, which vertices of $V_{\leq 2}$ of G' are in a minimal solution. Note that every vertex, that is in $V_{\leq 2}$ and also remains in G' , is on a maximal blue (red) path of G' . By the proof of Lemma 6 in [13], there are at most $16|S|$ maximal blue paths, also maximal red paths in G' . According to Lemma 1, $|V_{\leq 2}| \leq (16|S| + 1)^2$, and we can use exhausted search to find vertices in $V_{\leq 2} \cap S'$.

The total running time is $O^*(\sum_{t=0}^{|S|-1} \binom{28|S|+(16|S|)^2}{t}) = O^*(c^{|S|^2+|S|})$ for a constant c . ■

Obtaining Lemma 2, we are capable to enumerate all minimal solutions of DFVS by compact representations.

Theorem 3. *All minimal solutions of DFVS can be enumerated in time $O^*(c^{k^2+k})$ by compact representations for a constant c .*

Proof. We first find feedback vertex sets S_b and S_r of size at most $2k$ for G_b and G_r respectively, by the approximation algorithm of Bafna et al. [2]. Then $X = S_b \cup S_r$ is a dual feedback vertex set of size at most $4k$. Let X' be a minimal solution of size at most k . We guess which vertices of X are in X' , i.e., the vertex set $Y = X \cap X'$, and branch into $\sum_{|Y|=0}^k \binom{|X|}{|Y|}$ cases. For each case, $X - Y$ is a dual feedback vertex set of $G - Y$ and $X' - Y$ is a smaller dual feedback vertex set that is disjoint from $X - Y$. By Lemma 2, all possible sets $X' - Y$ can be enumerated in time $O^*(c_1^{|X|-|Y|+|X|-|Y|})$. Then combining Y and $X' - Y$, we get all minimal solutions of size at most k . The total running time for the algorithm is $O^*(\sum_{|Y|=0}^k \binom{|X|}{|Y|} c_1^{|X|-|Y|+|X|-|Y|})$ which is $O^*(c^{k^2+k})$ for a constant c as $|X| \leq 4k$ and $|Y| \leq k$. ■

4 Concluding remarks

In this paper, we established the fixed parameter tractability of DFVS, and provided two algorithms to enumerate all minimal solutions of DFVS by compact representations in FPT time. Furthermore, we established the fixed parameter tractability of MFVS by generalizing the algorithm of DFVS.

Note that there are at most hk set-vertices in the auxiliary graph H in the proof of Theorem 2, and each copy-vertex dominates at most h set-vertices. Thus there are at most $\binom{hk}{h}$ copy-vertices with different neighbors, which means that the number of copy-vertices can be up bounded by $\binom{hk}{h}$. Then we can use exhausted search of these copy-vertices to enumerate all minimal solutions of MFVS.

Theorem 4. *All minimal solutions of MFVS can be enumerated in FPT time by compact representations.*

For edge-bicolored graphs, instead of considering monochromatic cycles, we can study alternating cycles which are cycles alternates between blue and red edges. As suggested by Cai Leizhen, it is very interesting to consider the parameterized complexity of the following problem:

FEEDBACK VERTEX SET FOR ALTERNATING CYCLES (AFVS):

INPUT: Edge-bicolored graph G , parameter k .

QUESTION: Find a vertex set S of size at most k that hits all alternating cycles.

We observe that AFVS is a generalization of DIRECTED FEEDBACK VERTEX SET [8] which hits all cycles of a digraph. Given a digraph G , we can obtain an edge-bicolored graph G' by replacing each arc uv by a blue edge ux and a red edge xv . It is easy to check that G' has a feedback vertex set of size at most k for alternating cycles iff G has a directed feedback vertex set of size at most k .

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